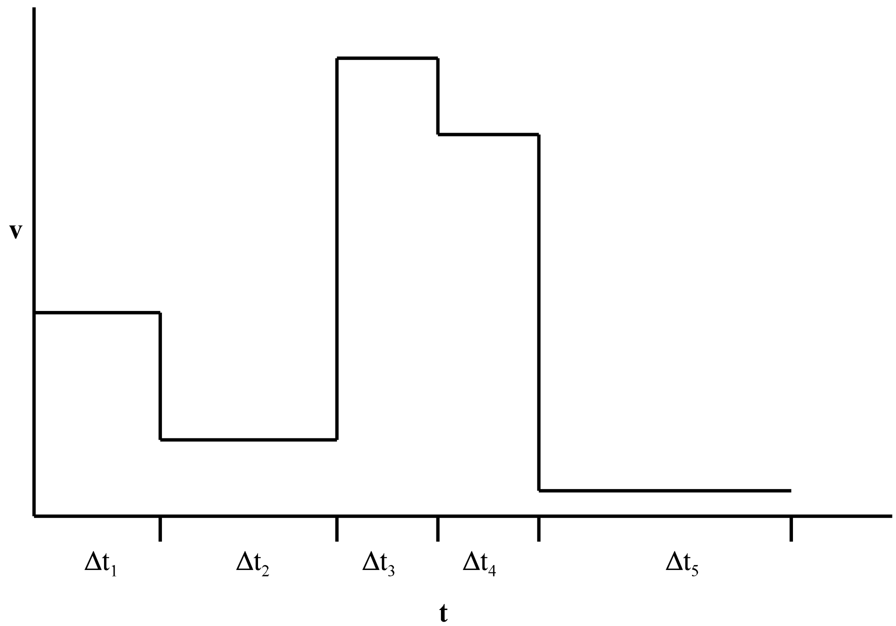
The Average Of A Continuously Changing Function

Suppose we want to determine the average speed of a car. The average speed can be determined by dividing the total distance traveled by the total time required.

$$v_{ave} = distance/time \qquad (1)$$

Consider the case in which the speed varies between intervals of time as represented by the graph below:



The total distance can be calculated by determining the distance traveled at each speed and then adding these distances together.

$$\begin{array}{ll} & \Delta x_1 = v_1 \Delta t_1 \\ & \Delta x_2 = v_2 \Delta t_2 \\ & \Delta x_3 = v_3 \Delta t_3 \\ & \quad . \qquad\quad . \\ & \quad . \qquad\quad . \\ + & \Delta x_n = v_n \Delta t_n \\ \hline & distance = \sum_i v_i \Delta t_i \end{array} \qquad (2)$$

Notice that this distance equals the area enclosed by the graph.

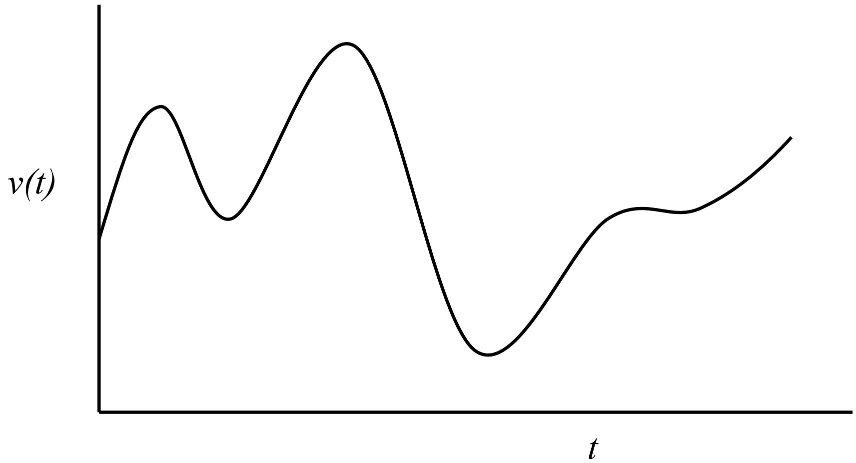
Since the total time is simply the sum of the time intervals, Δt_i,

$$time = \sum_i \Delta t_i \qquad (3)$$

and eq. 1 can be expressed as

$$v_{ave} = \frac{\sum v_i \Delta t_i}{\sum \Delta t_i} \qquad (4)$$

Now consider the case in which the speed changes instantaneously with time.



The total distance traveled is still equal to the area bounded by the graph and the horizontal axis. If the speed can be expressed as a function of time, v(t), the average speed can be calculated by the expression

$$v_{ave} = \frac{\int_0^t v(t)dt}{\int_0^t dt}$$

This approach to averaging is true in general and is not limited to time as the variable. In general, any integrable function, f(u), can be averaged over any interval from u_a to u_b by the expression

$$f_{ave} = \frac{\int_{u_a}^{u_b} f(u)du}{\int_{u_a}^{u_b} du} \qquad (5)$$

Household electricity is AC and varies sinusoidally with time so that the average current, I_{ave}, and the average voltage, V_{ave}, are zero. Electric power is the product of current and voltage. Does this mean that the average power consumed by a light bulb, for instance, is zero? Certainly, we know that the bulb uses power.

The paradox is resolved when we consider the average of the instantaneous power. Instantaneous power, p, is the product of instantaneous current, i, and instantaneous voltage, v, or

$$p = iv \tag{6}$$

If current flows through a bulb of constant resistance, R, Ohm's law can be used to express the instantaneous power as

$$p = \frac{v^2}{R} \tag{7}$$

and

$$p = i^2R \tag{8}$$

Eqs. 7 and 8 show that, unlike instantaneous voltage and instantaneous current, the instantaneous power is always greater than or equal to zero. Compare the three graphs below. Can you see how eq. 6 also relates the three graphs?

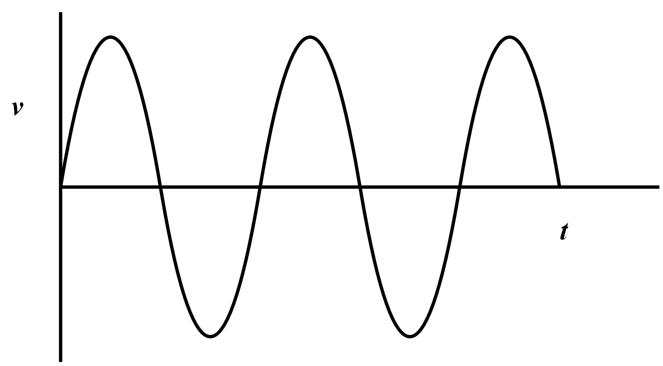


fig.1

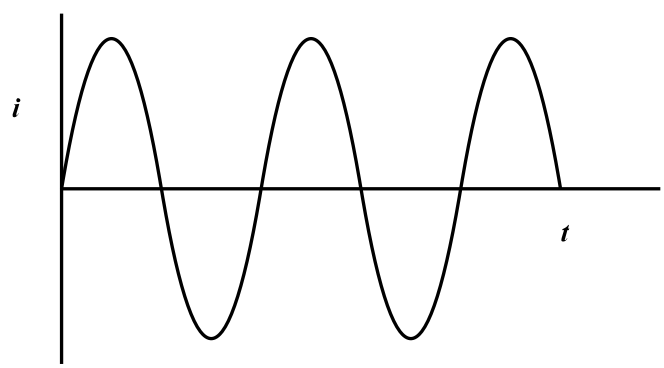


fig.2

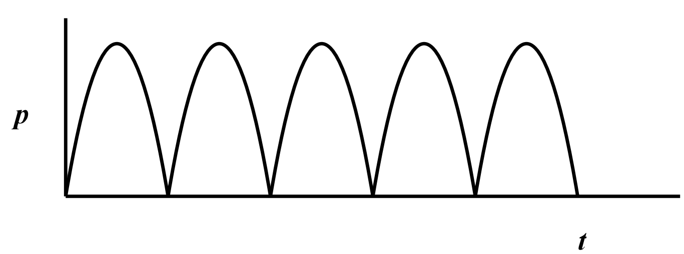


fig.3

It is evident from fig. 3, that the average power, P_{ave}, is not zero, even though the average voltage and the average current are zero. It is necessary to define a special kind of "average" for voltage and current that is meaningful in calculating average power. Effective voltage, V_{eff}, and effective current, I_{eff}, are the terms used to describe these special averages, and they satisfy the equation

$$P_{ave} = I_{eff} V_{eff} \qquad (9)$$

According to Ohm's law, eq. 9 can also be written as

$$P_{ave} = \frac{V_{eff}^2}{R} \qquad (10)$$

and

$$P_{ave} = I_{eff}^2 R \qquad (11)$$

Using eq. 5, the average power can be calculated by

$$P_{ave} = \frac{\int p(t)dt}{\int dt} \qquad (12)$$

If the AC is sinusoidal, the instantaneous voltage can be written as

$$v = V_{\max} \sin(\omega t) \qquad (13)$$

where $\omega = 2\pi/T$ and T is the period.

When eqs. 13 and 7 are substituted into eq. 12, the average power during one cycle can be determined by integrating over the period; and,

$$P_{ave} = \frac{V_{\max}^2}{RT} \int_0^T \sin^2(\omega t) dt \qquad (14)$$

or, upon integration,

$$P_{ave} = \frac{V_{\max}^2}{2R} \qquad (15)$$

Combining eqs. 10 and 15, we can express the effective voltage in terms of the maximum voltage:

$$V_{eff}^2 = \frac{V_{\max}^2}{2}$$

or

$$V_{eff} = \frac{V_{\max}}{\sqrt{2}} \approx .707\, V_{\max} \qquad (16)$$

A similar expression can be obtained for effective current. Using Ohm's law, eq. 13 can produce the time dependent equation for instantaneous current,

$$i = I_{\max}\sin(\omega t) \qquad (17)$$

Substituting eqs. 8 and 17 into eq. 12, we once again obtain an equation for average power; this time in terms of maximum current:

$$P_{ave} = \frac{I_{\max}^2 R}{2} \qquad (18)$$

Combining eqs. 11 and 18, we can express the effective current in terms of the maximum current;

$$I_{eff} = \frac{I_{\max}}{\sqrt{2}} \approx .707\, I_{\max} \qquad (19)$$

In perspective, notice that effective voltage and effective current are the results of first squaring the instantaneous voltages and currents, then finding their averages, and finally taking the square root. For this reason, the effective voltage and current are also referred to as the root-mean-square voltage and the root-mean-square current.

Not only are eqs. 16 and 19 important, but when substituted into eq. 9, produce the meaningful relationship between average power and maximum power,

$$P_{ave} = \frac{P_{\max}}{2} \qquad (20)$$

Using fig. 3, how would you interpret eq. 20 graphically?