

CHAPTER 2

NEWTONIAN MECHANICS: RECTILINEAR MOTION OF A PARTICLE

2.1 (a) $\ddot{x} = \frac{1}{m}(F_0 + ct)$

$$\dot{x} = \int_0^t \frac{1}{m}(F_0 + ct) dt = \frac{F_0}{m}t + \frac{c}{2m}t^2$$

$$x = \int_0^t \left(\frac{F_0}{m}t + \frac{c}{2m}t^2 \right) dt = \frac{F_0}{m}t^2 + \frac{c}{6m}t^3$$

(b) $\ddot{x} = \frac{F_0}{m} \sin ct$

$$\dot{x} = \int_0^t \frac{F_0}{m} \sin ct dt = -\frac{F_0}{cm} \cos ct \Big|_0^t = \frac{F_0}{cm}(1 - \cos ct)$$

$$x = \int_0^t \frac{F_0}{cm}(1 - \cos ct) dt = \frac{F_0}{cm} \left(1 - \frac{1}{c} \sin ct \right)$$

(c) $\ddot{x} = \frac{F_0}{m} e^{ct}$

$$\dot{x} = \frac{F_0}{cm} e^{ct} \Big|_0^t = \frac{F_0}{cm}(e^{ct} - 1)$$

$$x = \frac{F_0}{cm} \left(\frac{1}{c} e^{ct} - \frac{1}{c} - t \right) = \frac{F_0}{c^2 m} (e^{ct} - 1 - ct)$$

2.2 (a) $\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx} \cdot \frac{dx}{dt} = \dot{x} \frac{d\dot{x}}{dx}$

$$\dot{x} \frac{d\dot{x}}{dx} = \frac{1}{m}(F_0 + cx)$$

$$\dot{x} d\dot{x} = \frac{1}{m}(F_0 + cx) dx$$

$$\frac{1}{2} \dot{x}^2 = \frac{1}{m} \left(F_0 x + \frac{cx^2}{2} \right)$$

$$\dot{x} = \left[\frac{x}{m} (2F_0 + cx) \right]^{\frac{1}{2}}$$

(b) $\ddot{x} = \dot{x} \frac{d\dot{x}}{dx} = \frac{1}{m} F_0 e^{-cx}$

$$\dot{x}d\dot{x} = \frac{1}{m} F_0 e^{-cx} dx$$

$$\frac{1}{2} \dot{x}^2 = -\frac{F_0}{cm} (e^{-cx} - 1) = \frac{F_0}{cm} (1 - e^{-cx})$$

$$\dot{x} = \left[\frac{2F_0}{cm} (1 - e^{-cx}) \right]^{\frac{1}{2}}$$

(c) $\ddot{x} = \dot{x} \frac{d\dot{x}}{dx} = \frac{1}{m} (F_0 \cos cx)$

$$\dot{x}d\dot{x} = \frac{F_0}{m} \cos cx dx$$

$$\frac{1}{2} \dot{x}^2 = \frac{F_0}{cm} \sin cx$$

$$\dot{x} = \left(\frac{2F_0}{cm} \sin cx \right)^{\frac{1}{2}}$$

2.3 (a) $V(x) = -\int_{x_0}^x (F_0 + cx) dx = -F_0 x - \frac{cx^2}{2} + C$

(b) $V(x) = -\int_{x_0}^x F_0 e^{-cx} dx = \frac{F_0}{c} e^{-cx} + C$

(c) $V(x) = -\int_{x_0}^x F_0 \cos cx dx = -\frac{F_0}{c} \sin cx + C$

2.4 (a) $F(x) = -\frac{dV(x)}{dx} = -kx$

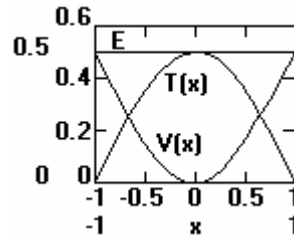
$$V(x) = \int_0^x kx dx = \frac{1}{2} kx^2$$

(b) $T_0 = T(x) + V(x)$

$$T(x) = T_0 - V(x) = \frac{1}{2} k(A - x^2)$$

(c) $E = T_0 = \frac{1}{2} kA^2$

(d) turning points @ $T(x_1) \rightarrow 0 \quad \therefore x_1 = \pm A$



2.5 (a) $F(x) = -kx + \frac{kx^3}{A^2}$ so $V(x) = \int_0^x \left(kx - \frac{kx^3}{A^2} \right) dx = \frac{1}{2} kx^2 - \frac{1}{4} \frac{kx^4}{A^2}$

(b) $T(x) = T_0 - V(x) = T_0 - \frac{1}{2} kx^2 + \frac{1}{4} \frac{kx^4}{A^2}$

(c) $E = T_0$

(d) $V(x)$ has maximum at $|F(x_m)| \rightarrow 0$

$$kx_m - \frac{kx_m^3}{A^2} = 0 \quad x_m = \pm A$$

$$V(x_m) = \frac{1}{2}kA^2 - \frac{1}{4}\frac{kA^4}{A^2} = \frac{1}{4}kA^2$$

If $E < V(x_m)$ turning points exist.

Turning points @ $T(x_1) \rightarrow 0$ let $u = x_1^2$

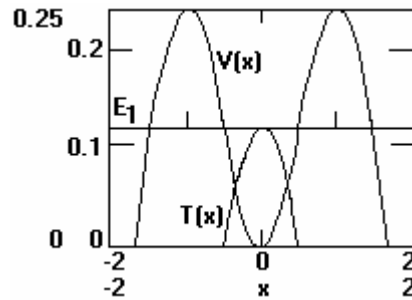
$$E - \frac{1}{2}ku + \frac{1}{4}\frac{ku^2}{A^2} = 0$$

solving for u , we obtain

$$u = A^2 \left[1 \pm \left(1 - \frac{4E}{kA^2} \right)^{\frac{1}{2}} \right]$$

or

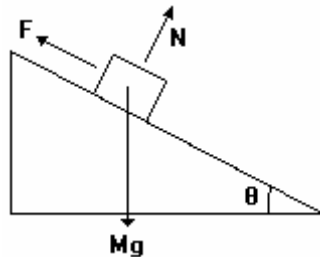
$$x_1 = \pm A \left[1 - \sqrt{\left(1 - \frac{4E}{kA^2} \right)} \right]^{\frac{1}{2}}$$



2.6 $\dot{x} = v(x) = \frac{\alpha}{x} \quad \ddot{x} = -\frac{\alpha}{x^2} \dot{x} = -\frac{\alpha^2}{x^3}$

$$F(x) = m\ddot{x} = -\frac{m\alpha^2}{x^3}$$

2.7



$$F \geq Mg \sin \theta$$

2.8 $F = m\ddot{x} = m\dot{x} \frac{d\dot{x}}{dx}$

$$\dot{x} = bx^{-3}$$

$$\frac{d\dot{x}}{dx} = -3bx^{-4}$$

$$F = m(bx^{-3})(-3bx^{-4})$$

$$F = -3mb^2x^{-7}$$

2.9 (a) $V = mgx = (.145kg) \left(9.8 \frac{m}{s^2} \right) (1250ft) \left(.3048 \frac{m}{ft} \right) = 541J$

(b) $T = \frac{1}{2}mv^2 = \frac{1}{2}mv_i^2 = \frac{1}{2}m \left(\frac{mg}{c_2} \right) = \frac{1}{2} \frac{m^2 g}{.22D^2}$

$$T = \frac{(.145kg)^2 \left(9.8 \frac{m}{s^2} \right)}{(2)(.22) \left[(2)(.0366) \right]^2 \frac{kg}{m}} = 87J$$

$$\begin{aligned} \int Fdx &= \int -cv^2 dx = -c \int v^3 dt = -c \int \left(-v_i \tanh \left(\frac{t}{\tau} \right) \right)^3 dt \\ &= cv_i^3 \tau \left[-\frac{1}{2} \tanh^2 \left(\frac{t}{\tau} \right) + \int \tanh \left(\frac{t}{\tau} \right) d \left(\frac{t}{\tau} \right) \right] \\ &= cv_i^3 \tau \left[-\frac{1}{2} \tanh^2 \left(\frac{t}{\tau} \right) + \ln \cosh \left(\frac{t}{\tau} \right) \right] \end{aligned}$$

Now $\tanh^2 \left(\frac{t}{\tau} \right) \cong 1$ for $t \gg \tau$

Meanwhile $x = \int v dt = \int \left(-v_i \tanh \left(\frac{t}{\tau} \right) \right) dt = v_i \tau \ln \cosh \left(\frac{t}{\tau} \right)$

$$\ln \cosh \left(\frac{t}{\tau} \right) = \frac{x}{v_i \tau}$$

$$x = (1250ft) \left(.3048 \frac{m}{ft} \right) = 381m$$

$$v_i = \left(\frac{mg}{c_2} \right)^{\frac{1}{2}} = \left[\frac{(.145kg) \left(9.8 \frac{m}{s^2} \right)}{(.22)(.0732)^2 \frac{kg}{m}} \right]^{\frac{1}{2}} = 34.72 \frac{m}{s}$$

$$\tau = \left(\frac{m}{c_2 g} \right)^{\frac{1}{2}} = \left[\frac{(.145kg)}{(.22)(.0732)^2 \frac{kg}{m} \left(9.8 \frac{m}{s^2} \right)} \right]^{\frac{1}{2}} = 3.543s$$

$$\int Fdx = (.22)(.0732)^2 (34.72)^3 (3.543) \left[-.5 + \frac{3.81}{(34.72)(3.54)} \right] = 454J$$

$$V - T = 541J - 87J = 454J$$

2.10 Going up: $F_x = -mg \sin 30^\circ - \mu mg \cos 30^\circ$

$$\ddot{x} = -g(\sin 30^\circ + 0.1 \cos 30^\circ) = -5.749 \frac{m}{s^2}$$

$$v = v_0 + at$$

at the highest point $v = 0$ so $t_{up} = -\frac{v_0}{a} = 0.174v_0 s$

$$x_{up} = v_0 t_{up} + \frac{1}{2} a t_{up}^2 = 0.174v_0^2 - .087v_0^2 = 0.087v_0^2 m$$

Going down: $x'_0 = 0.087v_0^2$, $v'_0 = 0$, $a' = -9.8(0.5 - 0.0866)$

$$x_{down} = 0 = 0.087v_0^2 - \frac{1}{2} 4.0513 t_{down}^2$$

$$t_{down} = 0.207v_0 s$$

$$t_{total} = t_{up} + t_{down} = 0.381v_0 s$$

2.11 $a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx} = -\frac{c}{m} v^{\frac{3}{2}}$

$$v^{-\frac{1}{2}} dv = -\frac{c}{m} dx$$

$$\int_{v_0}^v v^{-\frac{1}{2}} dv = \int_0^{x_{max}} -\frac{c}{m} dx$$

$$-2v_0^{\frac{1}{2}} = -\frac{c}{m} x_{max}$$

$$x_{max} = \frac{2mv_0^{\frac{1}{2}}}{c}$$

2.12 Going up: $F_x = -mg - c_2 v^2$

$$a = v \frac{dv}{dx} = -g - kv^2, \quad k = \frac{c^2}{m}$$

$$\int_{v_0}^v \frac{v dv}{-g - kv^2} = \int_0^x dx$$

$$-\frac{1}{2k} \ln(-g - kv^2) \Big|_{v_0}^v = x$$

$$\frac{g + kv^2}{g + kv_0^2} = e^{-2kx}$$

$$v^2 = \left(\frac{g}{k} + v_0^2 \right) e^{-2kx} - \frac{g}{k}$$

Going down: $F_x = -mg + c_2 v^2$

$$v \frac{dv}{dx} = -g + kv^2$$

$$\int_0^v \frac{v dv}{-g + kv^2} = \int_0^x dx$$

$$\frac{1}{2k} \ln(-g + kv^2) \Big|_0^v = x - x_0$$

$$1 - \frac{k}{g} v^2 = e^{2kx} e^{-2kx_0}$$

$$v^2 = \frac{g}{k} - \left(\frac{g}{k} e^{-2kx_0} \right) e^{2kx}$$

2.13 At the top $v = 0$ so $e^{-2kx_{\max}} = \frac{\frac{g}{k}}{\frac{g}{k} + v_0^2}$

Coming down $x_0 = x_{\max}$ and at the bottom $x = 0$

$$v^2 = \frac{g}{k} - \left(\frac{g}{k} \right)^2 \frac{1}{\left(\frac{g}{k} + v_0^2 \right)} (1) = \frac{\left(\frac{g}{k} \right) v_0^2}{\frac{g}{k} + v_0^2}$$

$$v = \frac{v_t v_0}{\left(v_t^2 + v_0^2 \right)^{\frac{1}{2}}}, \quad v_t = \sqrt{\frac{g}{k}} = \sqrt{\frac{mg}{c_2}}$$

2.14 $a = v \frac{dv}{dx} = -\frac{k}{m} x^{-2}$

$$\int_0^v v dv = \int_b^x -\frac{k dx}{mx^2}$$

$$\frac{1}{2} v^2 = \frac{k}{m} \left(\frac{1}{x} - \frac{1}{b} \right)$$

$$v = \frac{dx}{dt} = \left[\frac{2k}{m} \left(\frac{1}{x} - \frac{1}{b} \right) \right]^{\frac{1}{2}} = \left[\frac{2k}{mb} \left(\frac{b-x}{x} \right) \right]^{\frac{1}{2}}$$

$$\int_0^t dt = \int_b^0 \left[\frac{mb}{2k} \left(\frac{x}{b-x} \right) \right]^{\frac{1}{2}} dx = \left(\frac{mb^3}{2k} \right)^{\frac{1}{2}} \int_1^0 \left(\frac{\frac{x}{b}}{1 - \frac{x}{b}} \right)^{\frac{1}{2}} d\left(\frac{x}{b} \right)$$

Since $x \leq b$, say $\frac{x}{b} = \sin^2 \theta$

$$t = \left(\frac{mb^3}{2k} \right)^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^0 \frac{\sin \theta (2 \sin \theta \cos \theta d\theta)}{\cos \theta} = \left(\frac{2mb^3}{k} \right)^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^0 \sin^2 \theta d\theta$$

$$t = \left(\frac{mb^3}{8k} \right)^{\frac{1}{2}} \pi$$

2.15 $m \frac{dv}{dt} = mg - c_1 v - c_2 v^2$

$$\int_0^t \frac{dt}{m} = \int_0^v \frac{dv}{mg - c_1 v - c_2 v^2}$$

Using $\int \frac{dx}{a + bx + cx^2} = \frac{1}{\sqrt{b^2 - 4ac}} \ln \frac{2cx + b - \sqrt{b^2 - 4ac}}{2cx + b + \sqrt{b^2 - 4ac}},$

$$\frac{t}{m} = \frac{1}{\sqrt{c_1^2 + 4mgc_2}} \ln \frac{-2c_2 v - c_1 - \sqrt{c_1^2 + 4mgc_2}}{-2c_2 v - c_1 + \sqrt{c_1^2 + 4mgc_2}} \Big|_0^v$$

$$\frac{t}{m} (c_1^2 + 4mgc_2)^{\frac{1}{2}} = \ln \frac{(2c_2 v + c_1 + \sqrt{c_1^2 + 4mgc_2})(c_1 - \sqrt{c_1^2 + 4mgc_2})}{(2c_2 v + c_1 - \sqrt{c_1^2 + 4mgc_2})(c_1 + \sqrt{c_1^2 + 4mgc_2})}$$

as $t \rightarrow \infty$, $2c_2 v_t + c_1 - \sqrt{c_1^2 + 4mgc_2} = 0$

$$v_t = -\frac{c_1}{2c_2} + \left[\left(\frac{c_1}{2c_2} \right)^2 + \frac{mg}{c_2} \right]^{\frac{1}{2}}$$

Alternatively, when $v = v_t$,

$$m \frac{dv}{dt} = 0 = mg - c_1 v_t - c_2 v_t^2$$

$$v_t = -\frac{c_1}{2c_2} + \left[\left(\frac{c_1}{2c_2} \right)^2 + \frac{mg}{c_2} \right]^{\frac{1}{2}}$$

2.16 $c_1 = (1.55 \times 10^{-4})(10^{-2}) = 1.55 \times 10^{-6} \frac{kg}{s}$

$$c_2 = (0.22)(10^{-2})^2 = 2.2 \times 10^{-5} \frac{kg}{s}$$

$$v_t = -\frac{1.55 \times 10^{-6}}{2 \times 2.2 \times 10^{-5}} + \left[\left(\frac{1.55 \times 10^{-6}}{2 \times 2.2 \times 10^{-5}} \right)^2 + \frac{(10^{-7})(9.8)}{2.2 \times 10^{-5}} \right]^{\frac{1}{2}}$$

$$v_t = 0.179 \frac{m}{s}$$

Using equation 2.29, $v_t = \sqrt{\frac{(10^{-7})(9.8)}{2.2 \times 10^{-5}}} = 0.211 \frac{m}{s}$

2.17 $m \frac{dv}{dt} = mv \frac{dv}{dx} = f(x)g(v)$

$$\frac{mvdv}{g(v)} = f(x)dx$$

By integration, get $v = v(x) = \frac{dx}{dt}$

If $F(x, t) = f(x)g(t)$:

$$m \frac{d^2x}{dt^2} = m \frac{d}{dt} \left(\frac{dx}{dt} \right) = f(x)g(t)$$

This cannot, in general, be solved by integration.

If $F(v, t) = f(v)g(t)$:

$$m \frac{dv}{dt} = f(v)g(t)$$

$$\frac{mdv}{f(v)} = g(t)dt$$

Integration gives $v = v(t)$

$$\frac{dx}{dt} = v(t)$$

$$dx = v(t)dt$$

A second integration gives $x = x(t)$

2.18 $F = kv = m \frac{dv}{dt}$

$$kv = \frac{m}{v} \frac{dv}{dt} = m \frac{dv}{dx}$$

$$\int kv dx = \int m dv$$

$$k \frac{x^2}{2} = mv + C$$

At $x = 0, t = 0$

$$v = v_0$$

$$C = -mv_0$$

$$k \frac{x^2}{2} + mv_0 = m \frac{dx}{dt}$$

$$\int dt = \int \frac{dx}{kx^2/2m + v_0} = \int \frac{dx}{A^2 + B^2 x^2} \quad \text{where } A^2 = v_0 \quad B^2 = \frac{k}{2m}$$

$$t = \frac{1}{AB} \tan^{-1} \frac{B}{A} x$$

Solving ...

$$x = \left(\frac{2mv_0}{k} \right)^{\frac{1}{2}} \tan \left(\frac{kv_0}{2m} \right)^{\frac{1}{2}} t$$

2.19 $F(x) = -Ae^{\alpha x} = m\ddot{x}$ or $F(v) = -Ae^{\alpha v} = m\dot{v}$ $\frac{dv}{e^{\alpha v}} = -\frac{A}{m} dt$

Let $u = e^{\alpha v}$ $du = \alpha e^{\alpha v} dv$ $dv = \frac{du}{\alpha e^{\alpha v}} = \frac{du}{\alpha u}$ $\therefore \frac{du}{u^2} = -\frac{\alpha A}{m} dt$

Integrating

$$\frac{1}{u} - \frac{1}{u_0} = \frac{A}{m} \alpha t \quad \text{and substituting } e^{\alpha v} = u$$

(a) $v = v_0 - \frac{1}{\alpha} \ln \left[1 + \frac{A}{m} e^{\alpha v_0} \alpha t \right]$

(b) $t = T$ @ $v = 0$

$$\alpha v_0 = \ln \left[1 + \frac{A}{m} e^{\alpha v_0} \alpha T \right]$$

$$e^{\alpha v_0} = 1 + \frac{A}{m} e^{\alpha v_0} \alpha T \quad T = \frac{m}{\alpha A} [1 - e^{-\alpha v_0}]$$

(c) $v \frac{dv}{dx} = v = -\frac{A}{m} e^{\alpha v}$ $\frac{v dv}{e^{\alpha v}} = -\frac{A}{m} dx$

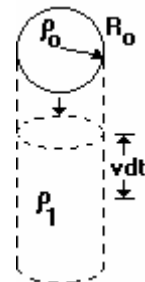
again, let $u = e^{\alpha v}$ $du = \alpha u dv$ or $dv = \frac{du}{\alpha u}$ $v = \frac{1}{\alpha} \ln u$

$$\frac{\left[\frac{1}{\alpha} \ln u \right] \frac{du}{\alpha u}}{u} = -\frac{A}{m} dx \quad \text{Integrating and solving}$$

$$x = \frac{m}{\alpha^2 A} [1 - (1 + \alpha v_0) e^{-\alpha v_0}]$$

2.20

$$F = \frac{d(mv)}{dt} = mv + vm = mg$$



but $m = \rho_0 \frac{4}{3} \pi r^3$ $m = \rho_1 \pi r^2 v$

so (1) $\frac{4}{3} \pi \rho_0 r^3 v + \pi \rho_1 r^2 v^2 = \frac{4}{3} \pi^2 = \frac{4}{3} \pi \rho_0 r^3 g$

Now $\frac{\rho_1}{\rho_0} \approx 10^{-3}$ so, second term is negligible-small

hence $v \approx g$ and $v \approx gt$ speed $\propto t$ but

$\dot{m} = \rho_0 4\pi r^2 \dot{r} = \rho_1 \pi r^2 v$ or $\dot{r} \cong \frac{1}{4} \frac{\rho_1}{\rho_0} v$ Hence $r \approx \frac{1}{4} \frac{\rho_1}{\rho_0} gt$ and rate of growth $\propto t$

The exact differential equation from (1) above is:

$$\frac{4}{3} \pi \rho_0 r \left| \frac{4 \rho_0}{\rho_1} \ddot{r} \right| + \pi \rho_1 \left| \frac{4 \rho_0 \dot{r}}{\rho_1} \right|^2 = \frac{4}{3} \pi \rho_0 r g$$

which reduces to: $\ddot{r} + \frac{3\dot{r}^2}{r} = \frac{\rho_1}{4\rho_0} g$

Using Mathcad, solve the above non-linear d.e. letting

$\frac{\rho_1}{\rho_0} \approx 10^{-3}$ and $R_0 \approx 0.01mm$ (small raindrop). Graphs

show that

$$v \propto \dot{r} \propto t \text{ and } r \propto t^2$$

